and let $V(x)$ be a positive-definite Liapunov function satisfying the condition $V_x{}^T F \equiv 0$. The control law

$$u = -\Lambda G^T V_x$$

quarantees the asymptotic stability of the system with respect to the variables $y_1 = \Psi_1(x), \ldots, y_r = \Psi_r(x)$ with the functional

$$J(u) = \frac{1}{2}\int_0^\infty [V_x{}^T G\Lambda G^T V_x + u^T \Lambda^{-1} u]\, dt$$

assuming its minimum value, provided that the function $V_x{}^T G\Lambda G^T V_x$ is positive-definite with respect to $y_1, \ldots, y_r$.

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Translated by L. K.

UDC 532.529.3

LAMINAR AXISYMMETRIC JET SUBMERGED IN A ROTATING FLUID

PMM Vol. 40, № 2, 1976, pp. 359-361

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(Received February 10, 1975)

A solution for a weakly nonself-similar axisymmetric jet submerged in a rotating viscous incompressible fluid is derived in a boundary layer approximation. An asymptotic expression is obtained for the jet field at considerable distances from the source, where it becomes self-similar.

1. Let a half-space filled by a viscous incompressible fluid and its solid plane boundary rotate at constant angular velocity ω around an axis normal to that plane.

We attach to the solid plane a right-hand system of cylindrical coordinates r, φ, z and make the half-space boundary to coincide with the plane $z = 0$ so that for every point of the fluid $z > 0$. Let us consider the problem of slow steady axisymmetric relative motions of the fluid in the half-space, induced by the velocity distribution at the solid plane

$$\mathbf{v}\,|_{z=0} = \mathbf{e}_z w_0(r) \tag{1.1}$$

with conditions at infinity

$$\mathbf{v} \to 0, \quad \mathbf{r} \to \infty \tag{1.2}$$

We shall consider the case when function $w_0(r)$ is finite and its three-dimensional scale L is considerably greater than the thickness of the Ekman boundary layer, i.e.

$$L \gg \sqrt{\nu / \omega} \tag{1.3}$$

Coriolis forces impede the motion of fluid in the direction normal to the axis of rotation and, according to the estimates presented in [1], when condition (1.3) is satisfied a boundary layer of the jet which spreads along the z-axis is formed. The system of linearized equations of the jet boundary layer in a rotating fluid in the axisymmetric case is of the form [1]

$$2\omega v = \frac{\partial h}{\partial r}, \quad 2\omega u = \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \right) \tag{1.4}$$

$$\frac{\partial h}{\partial z} = \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right), \quad \frac{\partial ru}{\partial r} + \frac{\partial rw}{\partial z} = 0, \quad h = \frac{p}{\rho} + \Phi$$

where u, v, w are velocity components of the fluid in the right-hand cylindrical system of coordinates attached to the rotating solid plane, p is the pressure, ρ the density and Φ is the potential of the centrifugal force field.

The suction of fluid into the jet induces a flow over the solid plane whose longitudinal scale is $\sim L$. This flow is directed along the normal to the axis of rotation and, consequently, the Coriolis and viscous forces are substantial in it. Equating the orders of magnitude of the latter we find that the scale of flow in the z-direction is $\sim \sqrt{\nu/\omega}$. Thus, when condition (1.3) is satisfied, a boundary layer is formed on the solid plane. In a linear approximation the velocity field for $z \lesssim \sqrt{\nu/\omega}$ represents the superposition of the jet field on the boundary layer at the solid plane. Since the z-component of velocity in the layer at that plane is appreciably smaller than in the jet field, it is possible to assume that approximately

$$w\,|_{z=0} = w_0(r) \tag{1.5}$$

where w is the z-component of velocity in the jet field.

The problem of determining the field of a weak jet submerged in a rotating fluid reduces to the integration of Eqs. (1.4) with boundary conditions (1.2) and (1.5).

2. The direct substitution into equations shows that system (1.4) admits particular solutions of the form

$$u = A\frac{\nu k^2}{2\omega} J_1(kr) e^{-\lambda z}, \quad v = -AJ_1(kr) e^{-\lambda z} \tag{2.1}$$

$$w = AJ_0(kr) e^{-\lambda z}, \quad h = A\frac{2\omega}{k} J_0(kr) e^{-\lambda z}, \quad \lambda = \frac{\nu k^3}{2\omega}$$

where A and k are parameters, and J_0 and J_1 are Bessel functions.

Let us construct the superposition

$$u = \frac{\nu}{2\omega}\int_0^\infty A(k) J_1(kr) e^{-\lambda z} k^3\, dk, \quad v = -\int_0^\infty A(k) J_1(kr) e^{-\lambda z} k\, dk \tag{2.2}$$

$$w = \int_0^\infty A(k) J_0(kr) e^{-\lambda z} k\, dk, \quad h = 2\omega \int_0^\infty A(k) J_0(kr) e^{-\lambda z}\, dk$$

of particular solutions (2.1), stipulating that the third formula in (2.2) must satisfy condition (1.5). As the result, we obtain an equation for $A(k)$. By reversing the Fourier-Bessel transformation we obtain

$$A(k) = \int_0^\infty w_0(r) J_0(kr) r dr \tag{2.3}$$

Integrals (2.2) are well convergent for $z > 0$ owing to the factor $e^{-\lambda z}$ appearing in the integrands. This makes it possible to obtain derivatives of expressions (2.2) with respect to r and z by differentiating under the integral sign. But then, by construction, formulas (2.2) and (2.3) yield a solution of system (1.4) in the half-space $z > 0$. It will be readily seen that this solution satisfies not only the boundary condition on plane $z = 0$ but also at infinity. Theoretically formulas (2.2) and (2.3) solve the boundary value problem defined by Eqs. (1.4) with boundary conditions (1.2) and (1.5).

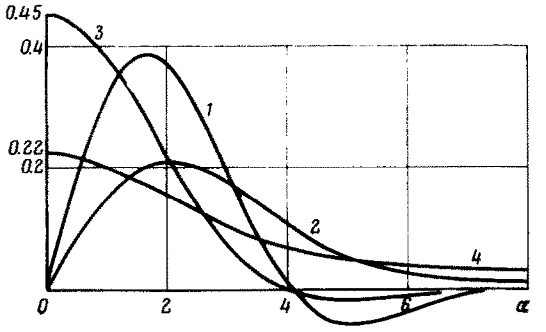

Fig. 1

3. Function $A(k)$, which is the Fourier-Bessel transformation of the finite function $w_0(r)$, can be expanded in a power series in the neighborhood of zero: $A(k) = ak^n + bk^{n+2} + \ldots$. According to [2] the asymptotic formula for the jet field can be obtained for $z \to \infty$ and fixed r from (2.2) by substituting ak^n for $A(k)$. It is of the form

$$u = \frac{\nu a}{2\omega} I_{13}, \quad v = -aI_{11}, \quad w = aI_{01}, \quad h = 2\omega a I_{00} \tag{3.1}$$

$$I_{pq} = \delta^{n+q+1} \int_0^\infty \varkappa^{n+q} J_p(\alpha\varkappa) e^{-\varkappa^3} d\varkappa$$

$$\delta = \left(\frac{2\omega}{\nu z}\right)^{1/3}, \quad \varkappa = \frac{k}{\delta}, \quad \alpha = \delta r$$

The adduced expressions show that the asymptotic formulas (3.1) yield a self-similar solution of system (1.4). The solution (3.1) is presented in Fig. 1 for $n = 0$; in this case $a = Q/2\pi$, where Q is the volume flow rate of jet sources. Curves *1—4* correspond to the following functions of α:

$$1 - \frac{5\pi z u}{Q\delta}, \quad 2 - \frac{-2\pi v}{Q\delta^2}, \quad 3 - \frac{2\pi w}{Q\delta^2}, \quad 4 - \frac{\pi h}{4\omega Q\delta}$$

The self-similar solution for the jet field for $n = 1$ was obtained in [1]. It should be noted, however, that by virtue of (2.3) function $A(k)$ is even, hence for $z \to \infty$ only self-similar integrals in (3.1) with even n can be asymptotic solutions of the boundary value problem (1.4), (1.2) and (1.5).

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